DEPARTMENT OF STATISTICS
UNIVERSITY OF BOTSWANA
STA 101: Mathematics for Business and Social Sciences 1

Date 06-11 2016 Time: 11-1.00PM Max: 100 Marks
Answer ALL questions within 120 minutes.

TEST 2 SOLUTIONS

Question 1

(a) Given $A=\begin{bmatrix}4 & 2\\ 9 & 5\end{bmatrix}$, find $\mathbf{AA^T}$.

(b) Let $B=\begin{bmatrix}4 & 1\\ 3 & c\end{bmatrix}$ Find c such that $|B|=5$

(c) Given that $A=\begin{pmatrix}4 & 2 & 0\\ 9 & 5 & 0\\ 0 & 0 & 3\end{pmatrix}$ find

(i) the determinant of A,
(ii) the inverse of A.

(5+5+5+10 = 25 Marks)

Solution 1

(a)

$$AA^T=\begin{bmatrix}4 & 2\\ 9 & 5\end{bmatrix}\begin{bmatrix}4 & 9\\ 2 & 5\end{bmatrix}=\begin{bmatrix}4.4+2.2 & 4.9+2.5\\ 9.4+5.2 & 9.9+5.5\end{bmatrix}=\begin{bmatrix}20 & 46\\ 46 & 106\end{bmatrix}$$

(b)

$|B| = 4 \times c - 1 \times 3$
$5 = 4c-3$
Thus $c=8/4=2$

(c)

(i)

(i) $\text{Det (B)} = 4\begin{vmatrix}5 & 0\\ 0 & 3\end{vmatrix}-2\begin{vmatrix}9 & 0\\ 0 & 3\end{vmatrix}+0\begin{vmatrix}9 & 5\\ 0 & 0\end{vmatrix}$ = 4(5.3-0.0)-2(9.3-0.0)+0 = 60-54+0 = 6

(ii) Inverse of A

Method 1 (Elementary row operation)

Consider the augmented matrix

$$\left[A|I_3\right]=\left[\begin{array}{ccc|ccc}4 & 2 & 0 & 1 & 0 & 0\\ 9 & 5 & 0 & 0 & 1 & 0\\ 0 & 0 & 3 & 0 & 0 & 1\end{array}\right]$$

$R1 \rightarrow (1/4)\, R1$

$$\left[A|I_3\right]=\left[\begin{array}{ccc|ccc}1 & 1/2 & 0 & 1/4 & 0 & 0\\ 9 & 5 & 0 & 0 & 1 & 0\\ 0 & 0 & 3 & 0 & 0 & 1\end{array}\right]$$

R2→R2-9R1
R3→(1/3) R3

$$\left[A\middle|I_3\right]=\left[\begin{array}{ccc|ccc}1 & 1/2 & 0 & 1/4 & 0 & 0\\ 0 & 1/2 & 0 & -9/4 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1/3\end{array}\right]$$

R2→2R2
R1→R1-(1/2)R2

$$\left[A\middle|I_3\right]=\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 5/2 & -1 & 0\\ 0 & 1 & 0 & -9/2 & 2 & 0\\ 0 & 0 & 1 & 0 & 0 & 1/3\end{array}\right]$$

$$A^{-1}=\begin{pmatrix}5/2 & -1 & 0\\ -9/2 & 2 & 0\\ 0 & 0 & 1/3\end{pmatrix}$$

Remark: The approach of elementary row operations is not unique. The number of row operations varies from individual to individual. Thus the above approach (row operations) is an indicative. However the solution is unique.

Method 2 (Cofactor method)

$$A^{-1}=\frac{adjA}{|A|}=\frac{C^T}{|A|}$$

Cofactor method

$$c_{11}=\begin{vmatrix}5 & 0\\ 0 & 3\end{vmatrix}=15 \quad c_{12}=-\begin{vmatrix}9 & 0\\ 0 & 3\end{vmatrix}=-27 \quad c_{13}=\begin{vmatrix}9 & 5\\ 0 & 0\end{vmatrix}=0$$

$$c_{21}=-\begin{vmatrix}2 & 0\\ 0 & 3\end{vmatrix}=-6 \quad c_{22}=\begin{vmatrix}4 & 0\\ 0 & 3\end{vmatrix}=12 \quad c_{23}=-\begin{vmatrix}4 & 2\\ 0 & 0\end{vmatrix}=0$$

$$c_{31}=\begin{vmatrix}2 & 0\\ 5 & 0\end{vmatrix}=0 \quad c_{32}=-\begin{vmatrix}4 & 0\\ 9 & 0\end{vmatrix}=0 \quad c_{33}=\begin{vmatrix}4 & 2\\ 9 & 5\end{vmatrix}=4.5-9.2=20-18=2$$

$$C=\begin{pmatrix}15 & -27 & 0\\ -6 & 12 & 0\\ 0 & 0 & 2\end{pmatrix}; \quad C^T=\begin{pmatrix}15 & -6 & 0\\ -27 & 12 & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$A^{-1}=\frac{1}{|A|}C^T=\frac{1}{6}\begin{pmatrix}15 & -6 & 0\\ -27 & 12 & 0\\ 0 & 0 & 2\end{pmatrix}=\begin{pmatrix}5/2 & -1 & 0\\ -9/2 & 2 & 0\\ 0 & 0 & 1/3\end{pmatrix}$$

Question 2

b) Mr Modise has 20 hectares for planting black-eye beans (x) and millet (y). The farmer has to decide how much of each to grow. The cost per hectare for black-eye beans is P30 and for millet is P20. The farmer has budgeted for P480. Black-eye beans require 1 man-day per hectare and millet requires 2 man-days per hectare. There are 36 man-days available. The profit on black-eye beans is P100 per hectare and on millet is P120 per hectare.

i) Formulate the farmer's crop production problem as a linear programming problem.

ii) Draw a graph showing the feasible region.

iii) Using the graph, find the number of hectares of each crop the farmer should sow to maximize profit. What is the profit at this production level?

(10+10+5 = 25 Marks)

Solution 2

Given:

Table of resources

	Black eye crop	Millet	Resources available
Cultivation cost/acre	P30	P20	P480
Labour /acre	1	2	36
Profit/acre	P100	P120	___

i) Formulation of L.P.P using the given information

Let x be no of acres of black eye crop sown and y be that of millet.

The objective function is : Maximize $P = 100x + 120y$

Subject to:

$$x + y \le 20$$
$$30x + 20y \le 480$$
$$x + 2y \le 36$$

and $x \ge 0, y \ge 0.$

ii) Graph showing the feasible region i.e. to represent the constraints of the L.P.P. on the (x-y) plane using a graph.

Plot the regions:

Equation (1) : $x + y = 20$

x	y
0	20
20	0

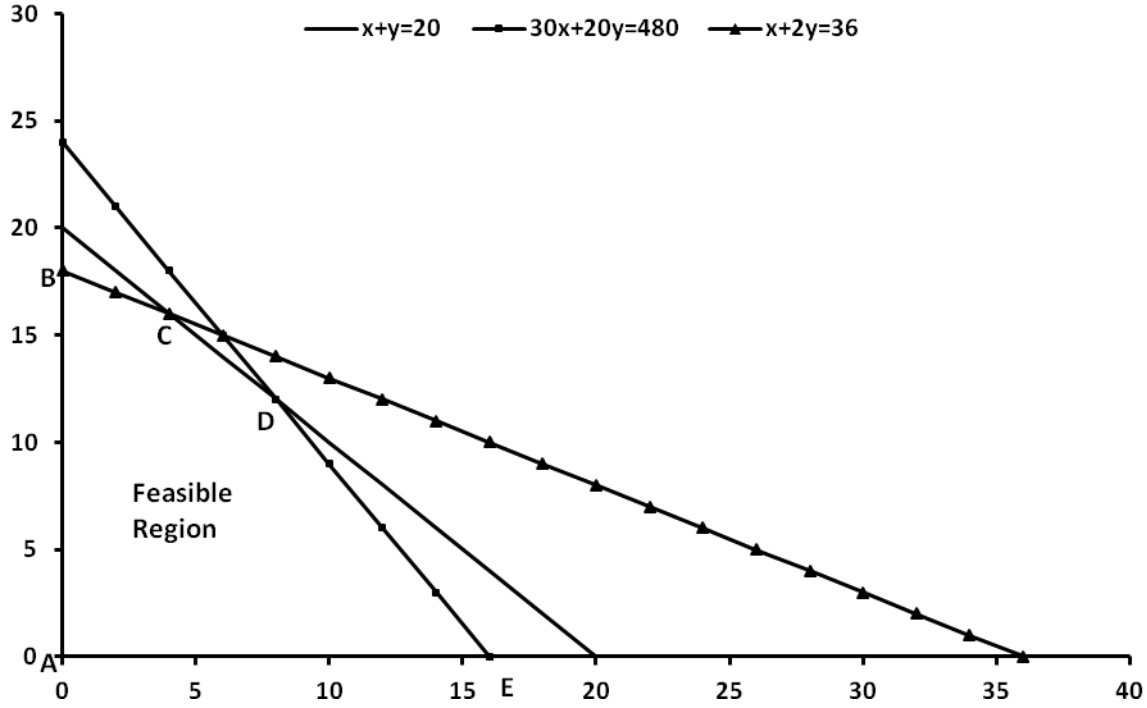
Equation (2) : $30x + 20y = 480$

x	y
0	24
16	0

Equation (3) : $x + 2y = 36$

x	y

0	18
36	0



iii) Evaluate the value of the objective function at these vertices

Corner Points:

Vertices	**Coordinates**	**P=100x+120y**
A	(0,0)	0
B	(0,18)	2160
C	(4,16)	2320
D	(8,12)	2240
E	(16,0)	1600

From the above table we see that P is maximized at the vertex C (4, 16) with the maximum profit P=2360.

Question 3

(a) Find x and y given the following equations

$x + 2y = 5$

$3x + 7y = 18$

(b) Solve the following system of linear equations by any method of your choice.

$x_1 - 2x_2 - x_3 = 3$

$3x_1 - 6x_2 - 5x_3 = 3$

$2x_1 - x_2 + x_3 = 0$

(7+18 = 25 Marks)

Solution 3

(a)

$$x + 2y = 5 \quad (1)$$
$$3x + 7y = 18 \quad (2)$$

You can either eliminate x or y. Let us eliminate x.

$3 \times (1)$ $\quad 3x + 6y = 15 \quad (3)$

$(2)-(3)$ $\quad y = 3$

Substitute $y=3$ in either of equation (1) and (2), let us substitute $y=3$ *in (1),* we have

$x + 2(3) = 5$, this implies as $x = 5-6 = -1$

Hence the solution for the system of equations (1) and (2) is $x = -1$ and $y = 3$.

(b)

(i) Elimination Method

We have

$$x_1 - 2x_2 - x_3 = 3 \quad (1)$$
$$3x_1 - 6x_2 - 5x_3 = 3 \quad (2)$$
$$2x_1 - x_2 + x_3 = 0 \quad (3)$$

You can either eliminate x_1 , x_2 or x_3 from any two equations.

Consider equations (1) and (2) let us eliminate x_1

$3 \times (1)$ $\quad 3x_1 - 6x_2 - 3x_3 = 9 \quad (4)$

(2) $\quad 3x_1 - 6x_2 - 5x_3 = 3 \quad (5)$

$(4)-(5)$ $\quad 2x_3 = 6 \rightarrow x_3 = 3 \quad (6)$

Here to note that x_2 is also eliminated.

Consider equations (1) and (3) and eliminate x_1

$2 \times (1)$ $\quad 2x_1 - 4x_2 - 2x_3 = 6 \quad (7)$

(3) $\quad 2x_1 - x_2 + x_3 = 0 \quad (8)$

$(7) - (8)$ $\quad -3x_2 - 3x_3 = 6 \quad (9)$

Now, substitute the value of $x_3 = 3$ from (6) in (9), we have

$$-3x_2 - 3(3) = 6 \quad \rightarrow x_2 = -5 \quad (10)$$

Finally, substitute the value of $x_3 = 3$ from (6) and $x_2 = -5$ from (10) in (3), we have

$$2x_1 - (-5) + 3 = 0 \quad \rightarrow x_1 = -4 \quad (11)$$

Hence the solution of system of linear equations is $x_1 = -4$; $x_2 = -5$ and $x_3 = 3$

(ii) The Gauss-Jordan Method

We have

$$x_1 - 2x_2 - x_3 = 3 \quad (1)$$
$$3x_1 - 6x_2 - 5x_3 = 3 \quad (2)$$
$$2x_1 - x_2 + x_3 = 0 \quad (3)$$

The augmented matrix is $\begin{bmatrix} 1 & -2 & -1 & :3 \\ 3 & -6 & -5 & :3 \\ -1 & 1 & 2 & :0 \end{bmatrix}$

R1→R2-3R1
R3→R3-2R1

$$\begin{bmatrix} 1 & -2 & -1 & :3 \\ 0 & 0 & -2 & :-6 \\ 0 & 3 & 3 & :-6 \end{bmatrix}$$

R2→R2/(-2)
R3→R3/3

$$\begin{bmatrix} 1 & -2 & -1 & :3 \\ 0 & 0 & 1 & :3 \\ 0 & 1 & 1 & :-2 \end{bmatrix}$$

R1→R1+2R3
R2→R2+R3

$$\begin{bmatrix} 1 & 0 & 1 & :-1 \\ 0 & 1 & 2 & :1 \\ 0 & 0 & -1 & :-3 \end{bmatrix}$$

R1→R1+R3
R2→R2+R3
R3→-R3

$$\begin{bmatrix} 1 & 0 & 0 & :-4 \\ 0 & 1 & 0 & :-5 \\ 0 & 0 & 1 & :3 \end{bmatrix}$$

The final matrix is the augmented matrix of the system $\left.\begin{matrix} x_1 = -4 \\ x_2 = -5 \\ x_3 = 3 \end{matrix}\right\}$ whose solution is (-4,-5, 3) and since the original system is equivalent to the final system, it follows that (-4,-5, 3) is also the solution of the original system.

Remark: The approach of Gauss-Jordan Method (elementary row operations) is not unique. The number of row operations varies from individual to individual. Thus the above approach (row operations) is an indicative. However the solution is unique.

(iv) Matrix Inversion Method

Solution

We have

$$x_1 - 2x_2 - x_3 = 3 \quad (1)$$
$$3x_1 - 6x_2 - 5x_3 = 3 \quad (2)$$
$$2x_1 - x_2 + x_3 = 0 \quad (3)$$

In matrix notation we have

$$\begin{bmatrix} 1 & -2 & -1 \\ 3 & -6 & -5 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

$$A \qquad X = b$$

AX=b

$A^{-1}AX=A^{-1}b \rightarrow IX= A^{-1}b \rightarrow X= A^{-1}b$ (4)

To get the solution, we have to find the inverse of A and pre-multiply by column vector b.

The inverse of A by cofactor method

$$A^{-1} = \frac{1}{\det(A)} adjA = \frac{1}{\det(A)} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T$$

$$\det(A) = [1(-6.1 - (-1).(-5)) + 2(3.1 - 2.(-5)) - 1(3.(-1) - 2.(-6)]$$
$$\det(A) = [-11 + 26 - 9] = 6$$

Cofactors/Ad joint

$$c_{11} = \begin{vmatrix} -6 & -5 \\ -1 & 1 \end{vmatrix} = -6 - 5 = -11;$$

$$c_{12} = -\begin{vmatrix} 3 & -5 \\ 2 & 1 \end{vmatrix} = -(3 + 10) = -13;$$

$$c_{13} = \begin{vmatrix} 3 & -6 \\ 2 & -1 \end{vmatrix} = -3 + 12 = 9;$$

$$c_{21} = -\begin{vmatrix} -2 & -1 \\ -1 & 1 \end{vmatrix} = -(-2 - 1) = 3;$$

$$c_{22} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = (1 + 2) = 3;$$

$$c_{23} = -\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = -(-1 + 4) = -3;$$

$$c_{31} = \begin{vmatrix} -2 & -1 \\ -6 & -5 \end{vmatrix} = (10-6) = 4;$$

$$c_{32} = -\begin{vmatrix} 1 & -1 \\ 3 & -5 \end{vmatrix} = -(-5+3) = 2;$$

$$c_{33} = \begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = (-6+6) = 0;$$

$$A^{-1} = \frac{1}{\det(A)} adjA = \frac{1}{6}\begin{bmatrix} -11 & -13 & 9 \\ 3 & 3 & -3 \\ 4 & 2 & 0 \end{bmatrix}^T = \frac{1}{6}\begin{bmatrix} -11 & 3 & 4 \\ -13 & 3 & 2 \\ 9 & -3 & 0 \end{bmatrix}$$

$X = A^{-1}b$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -11 & 3 & 4 \\ -13 & 3 & 2 \\ 9 & -3 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -33+9+0 \\ -39+9+0 \\ 27-9+0 \end{bmatrix} = \begin{bmatrix} -24/6 \\ -30/6 \\ 18/6 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 3 \end{bmatrix}$$

Thus the solution is x_1=*-4,* x_2=*-5* and x_3=*3*

Question 4

(a) Suppose 4.5,-13.5 and 40.5 are the fifth, sixth and seventh term of certain sequence respectively.

i) Find the first term of the sequence.

ii) Hence or otherwise find the tenth term a_{10} of the sequence.

(b) A new school opens with an enrollment of 20 students. At the beginning of very semester 5 additional students are enrolled.

i) How many students would the school have during the 1st, 2nd and 3rd semesters?

ii) How many students would have been enrolled into the school 10 years later give that there are two semesters in a year?

(7+6+5+7=25 Marks)

Solution 4

(a)

i) The n-th term of geometric sequence is $a_n = a_1 r^{n-1}$

Given,

$a_5 = 4.5$, $a_6 = -13.5$ and $a_7 = 40.5$

Method 1

$a_5 = 4.5 = a_1 (-3)^{5-1}$

Thus $a_1 = 4.5/81 = 0.0556$

Method 2

$a_{i-1} = a_i / r$

$a_4 = 4.5/-3 = -1.5$

$a_3 = -1.5/-3 = 0.5$

$a_2 = 0.5/-3 = -0.1667$

$a_1 = -0.1667/-3 = 0.0556$

(ii)

Method 1

$a_n = a_1 r^{n-1}$

$a_{10} = 0.0556 (-3)^{10-1} = 0.0556 \text{ x } (-19683) = -1094.38$ (because of rounding error different from Method 2)

Method 2

$a_i = a_{i-1}. r$

$a_8 = 40.5 (-3) = -121.5$

$a_9 = -121.5 (-3) = 364.5$

$a_{10} = 364.5 (-3) = -1093.5$

(b)

(i)The students during the 1st, 2nd and 3rd semesters are

$a_1 = 20; \quad a_2 = 25; \quad a_3 = 30$

$(ii)\ S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right)$

Given,

$a_1 = 20; \quad and \ \ d = 5$

$n = 10x2 = 20 \quad semesters$

$S_{20} = \frac{20}{2}\left(2x20 + 19x5\right) = 10\left(40 + 95\right)$

$= 10(135) = 1350 \ students \ at \ 10 \ years \ later$